# Permutations and Combinations

Finite Math

16 April 2019

# Factorial

There are often situations in which we have to multiply many consecutive numbers together, e.g., in problems of the form “from a pool of 8 letters, make words consisting of 5 letters without any repetition.” There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ of these.

# Factorial

There are often situations in which we have to multiply many consecutive numbers together, e.g., in problems of the form “from a pool of 8 letters, make words consisting of 5 letters without any repetition.” There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ of these. Let’s define a notation that will simplify writing these quantities down:

# Factorial

There are often situations in which we have to multiply many consecutive numbers together, e.g., in problems of the form “from a pool of 8 letters, make words consisting of 5 letters without any repetition.” There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ of these. Let’s define a notation that will simplify writing these quantities down:

## Definition (Factorial)

*For a natural number n,*

$$\begin{aligned} n! &= n(n-1)(n-2)\cdot\dots\cdot 2\cdot 1 \\ 0! &= 1 \end{aligned}$$

# Factorial

From this definition, we can see that

$$n! = n \cdot (n-1)! = n(n-1) \cdot (n-2)! = \cdots ,$$

that is, we can explicitly write out as many of the largest numbers as we need, then write the rest as a smaller factorial.

# Factorial

From this definition, we can see that

$$n! = n \cdot (n-1)! = n(n-1) \cdot (n-2)! = \cdots ,$$

that is, we can explicitly write out as many of the largest numbers as we need, then write the rest as a smaller factorial. For example, we could write

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!$$

if we wanted to bring special attention to 10 through 7.

# Example

## Example

*Find*

(a) $6!$

(b) $\dfrac{10!}{9!}$

(c) $\dfrac{10!}{7!}$

(d) $\dfrac{5!}{0!3!}$

(e) $\dfrac{20!}{3!17!}$

# Now You Try It!

## Example

*Find*

(a) $7!$

(b) $\dfrac{8!}{4!}$

(c) $\dfrac{8!}{4!(8-4)!}$

# Permutations

Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

# Permutations

Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

**Definition (Permutation)**

*A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition.*

# Permutations

Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

**Definition (Permutation)**

*A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition.*

In the problem above, we have 5 people, and 5 seats to fill.

# Permutations

Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

**Definition (Permutation)**

*A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition.*

In the problem above, we have 5 people, and 5 seats to fill. If we fill in the seats from left to right, we can put one of 5 people in the first, one of the remaining 4 in the second, one of 3 in the third, one of 2 in the fourth, and then there is only one person left to fill the fifth seat.

# Permutations

Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

**Definition (Permutation)**

*A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition.*

In the problem above, we have 5 people, and 5 seats to fill. If we fill in the seats from left to right, we can put one of 5 people in the first, one of the remaining 4 in the second, one of 3 in the third, one of 2 in the fourth, and then there is only one person left to fill the fifth seat. Using the multiplication principle, we see that there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

possible arrangements, or permutations.

# Permutations

**Theorem (Permutations of $n$ Objects)**

*The number of permutations of n distinct objects without repetition, denoted by ${}_nP_n$, is*

$${}_nP_n = n(n-1)\cdots 2\cdot 1 = n!.$$

# Permutations of Subsets

Sometimes we don't want to use all of the available options, such as when we're making 5 letter words without repetition out of a pool of 8 letters.

# Permutations of Subsets

Sometimes we don't want to use all of the available options, such as when we're making 5 letter words without repetition out of a pool of 8 letters.

**Definition (Permutation of $n$ Objects Taken $r$ at a Time)**

*A permutation of a set of $n$ distinct objects taken $r$ at a time without repetition is an arrangement of $r$ of the $n$ objects in a specific order.*

# Deriving a Formula for Permutations

If we have $n$ things, and we want to create a permutation using $r$ of them we have: $n$ choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the $r^{th}$ slot.

# Deriving a Formula for Permutations

If we have $n$ things, and we want to create a permutation using $r$ of them we have: $n$ choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the $r^{th}$ slot. This gives us

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

possible permutations.

# Deriving a Formula for Permutations

If we have $n$ things, and we want to create a permutation using $r$ of them we have: $n$ choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the $r^{th}$ slot. This gives us

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

possible permutations. We can also write out the product of all numbers $n$ though 1, then divide out the stuff we don't want: $n-r$ though 1 by using factorials

# Deriving a Formula for Permutations

If we have $n$ things, and we want to create a permutation using $r$ of them we have: $n$ choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the $r^{th}$ slot. This gives us

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

possible permutations. We can also write out the product of all numbers $n$ though 1, then divide out the stuff we don't want: $n-r$ though 1 by using factorials

$$\#\text{ of perm.} \quad = \quad n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

# Deriving a Formula for Permutations

If we have $n$ things, and we want to create a permutation using $r$ of them we have: $n$ choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the $r^{th}$ slot. This gives us

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

possible permutations. We can also write out the product of all numbers $n$ though 1, then divide out the stuff we don't want: $n-r$ though 1 by using factorials

$$\begin{aligned} \text{\# of perm.} &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\ &= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)(n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)(n-r-1) \cdots 2 \cdot 1} \end{aligned}$$

# Deriving a Formula for Permutations

If we have $n$ things, and we want to create a permutation using $r$ of them we have: $n$ choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the $r^{th}$ slot. This gives us

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

possible permutations. We can also write out the product of all numbers $n$ though 1, then divide out the stuff we don't want: $n-r$ though 1 by using factorials

$$\begin{aligned}
\text{\# of perm.} &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)(n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)(n-r-1) \cdots 2 \cdot 1} \\
&= \frac{n!}{(n-r)!}
\end{aligned}$$

# Permutations of Subsets

**Theorem (Number of Permutations of $n$ Objects Taken $r$ at a Time)**

*The number of permutations of n distinct objects taken r at a time without repetition is given by*

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

# Example

## Example

*Given the set $\{A, B, C, D\}$, how many permutations are possible for this set of 4 objects taken 2 at a time?*

# Now You Try It!

**Example**

*Find the number of permutations of* 30 *objects taken* 4 *at a time.*

# Now You Try It!

## Example

*Find the number of permutations of* 30 *objects taken* 4 *at a time.*

## Solution

$$ {}_{30}P_4 = \frac{30!}{(30-4)!} = \frac{30!}{26!} = 30 \cdot 29 \cdot 28 \cdot 27 = 657,720 $$

# Combinations

Suppose there is a bag that has 10 jelly beans, each with a different flavor. How many different combinations of 3 flavors can you draw from the bag?

# Combinations

Suppose there is a bag that has 10 jelly beans, each with a different flavor. How many different combinations of 3 flavors can you draw from the bag?

**Definition (Combinations)**

*A combination of a set of $n$ distinct objects taken $r$ at a time without repetition is an $r$-element subset of the set of $n$ objects. The arrangement of the elements in the subset does not matter.*

# Deriving a Formula for Combinations

If we have $n$ objects, and we wanted permutations of $r$ objects at a time, we could think of that as happening in two steps:

# Deriving a Formula for Combinations

If we have $n$ objects, and we wanted permutations of $r$ objects at a time, we could think of that as happening in two steps:

- Step 1: Choose the $r$ elements from among the $n$. This is the number of combinations that we are looking for, and we will call this number ${}_nC_r$.

# Deriving a Formula for Combinations

If we have $n$ objects, and we wanted permutations of $r$ objects at a time, we could think of that as happening in two steps:

- Step 1: Choose the $r$ elements from among the $n$. This is the number of combinations that we are looking for, and we will call this number ${}_nC_r$.
- Step 2: Put the $r$ elements into a specific order. This is just a permutation of $r$ elements, so there are $r!$ ways to do this.

# Deriving a Formula for Combinations

If we have $n$ objects, and we wanted permutations of $r$ objects at a time, we could think of that as happening in two steps:

- Step 1: Choose the $r$ elements from among the $n$. This is the number of combinations that we are looking for, and we will call this number ${}_nC_r$.
- Step 2: Put the $r$ elements into a specific order. This is just a permutation of $r$ elements, so there are $r!$ ways to do this.

Thus, using the multiplication principle, we can see that the number of permutations of $n$ objects taken $r$ at a time is

$$ {}_nP_r = {}_nC_r \cdot r! $$

# Deriving a Formula for Combinations

If we have $n$ objects, and we wanted permutations of $r$ objects at a time, we could think of that as happening in two steps:

- Step 1: Choose the $r$ elements from among the $n$. This is the number of combinations that we are looking for, and we will call this number ${}_nC_r$.
- Step 2: Put the $r$ elements into a specific order. This is just a permutation of $r$ elements, so there are $r!$ ways to do this.

Thus, using the multiplication principle, we can see that the number of permutations of $n$ objects taken $r$ at a time is

$$ {}_nP_r = {}_nC_r \cdot r! $$

So, we can solve for ${}_nC_r$ to get

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}. $$

# Combinations

**Theorem (Number of Combinations of $n$ Objects Taken $r$ at a Time)**

*The number of combinations of n distinct objects taken r at a time without repetition is given by*

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

# Example

## Example

*Form a committee of 12 people.*

(a) *In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person cannot hold more than one position?*

(b) *In how many ways can we choose a subcommittee of 4 people?*

# Another Example

**Example**

*Find the number of combinations of* 30 *objects taken* 4 *at a time.*

# Now You Try It!

## Example

*How many ways can a 3-person subcommittee be selected from a committee of 7 people? How many ways can a president, vice-president, and secretary be chosen from a committee of 7 people?*

# Now You Try It!

## Example

*How many ways can a 3-person subcommittee be selected from a committee of 7 people? How many ways can a president, vice-president, and secretary be chosen from a committee of 7 people?*

## Solution

*35; 210*

# Now You Try It!

## Example

*Find the number of combinations of* 67 *objects taken* 5 *at a time.*

# Now You Try It!

## Example

*Find the number of combinations of* 67 *objects taken* 5 *at a time.*

## Solution

*9,657,648*

# Poker Hands!

## Example

*Suppose we ave a standard 52-card deck and we are considering 5-card poker hands.*

(a) *How many hands have 3 hearts and 2 spades?*

(b) *How many hands have all the same suit? (I.e., what is the number of different flushes?)*

(c) *How many possible pairs are there? (The other three cards have a different number from the pair and each other.)*

(d) *How many possible 3 of a kinds are there? (The other two cards have a different number from the 3 of a kind and from each other.)*

(e) *How many full houses are possible? (A full house consists of a three of a kind and a pair, each from a different number.)*